

УДК 513.83; 517.5

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ON THE SHADOW PROBLEM FOR DOMAINS IN THE EUCLIDEAN SPACES

ЗАДАЧА ПРО ТІНЬ ДЛЯ ОБЛАСТЕЙ В ЕВКЛІДОВИХ ПРОСТОРАХ

In the present work, the problem about shadow, generalized on domains of space \mathbb{R}^n , $n \leq 3$, is investigated. Here the shadow problem means to find the minimal number of balls satisfying some conditions an such that every line passing through the given point intersects at least one ball of the collection. It is proved that to generate the shadow at every given point of any domain of the space \mathbb{R}^3 (\mathbb{R}^2) with collection of mutually non-overlapping closed or open balls which do not hold the point and with centers on the boundary of the domain, it is sufficient to have four balls.

В роботі досліджується задача про тінь, узагальнена на області простору \mathbb{R}^n , $n \leq 3$. Під задачею про тінь тут мається на увазі знаходження мінімального числа куль, що задовільняють деяким умовам, і таких, що кожна пряма, що проходить через дану точку, перетне принаймні одну кулю з набору. Доведено, що для того, щоб створити тінь в кожній даній точці довільної області простору \mathbb{R}^3 (\mathbb{R}^2) набором замкнених або відкрити куль, які попарно не перетинаються, не містять дану точку та з центрами на межі області, достаточо чотирьох (двох) таких куль.

В работе исследуется задача о тени, обобщенная на области пространства \mathbb{R}^n , $n \leq 3$. Под задачей о тени подразумевается нахождение минимального количества шаров, удовлетворяющих некоторым условиям, и таких, что каждая прямая, проходящая через заданную точку, пересечет хотя бы один шар из набора. Доказано, что для того, чтобы создать тень в каждой заданной точке произвольной области пространства \mathbb{R}^3 (\mathbb{R}^2) набором замкнутых или открытых шаров, попарно не пересекающихся, не содержащих заданную точку и с центрами на границе области, достаточно четырех (двух) таких шаров.

Introduction

In 1982 G. Khudaiberganov proposed the problem about shadow [8] that can be formulated as follows: *what is the minimal number of closed (open) non-overlapping balls in the n -dimensional Euclidean space with radiuses less than the sphere radius, with centers on it and such that every line passing through the center of the sphere would intersect at least one of the balls.*

This problem (and similar to it) may be written briefly as: *what minimal number of some objects generates the shadow at the point.*

This problem was solved by G. Khudaiberganov for the case $n = 2$: it was proved that two balls are sufficient for a circumference on the plane. For all that, it was also made the assumption that for the case $n > 2$ the minimal number of such balls is exactly equal to n .

Yu. Zelinskii mentioned this problem in many of his works (cf., e.g. [1-2]). The problem is also interesting from the standpoint of convex analysis as a special case of the problem about membership of a point to the 1-hull of the union of some collection of balls. In [4], Yu. Zelinskii and his students proved that three balls are not sufficient for the case $n = 3$, but it is possible to generate the shadow at the center of the sphere with four balls. In their work it is also proved that for the general case the minimal number is $n + 1$ balls. Thus, G. Khudaiberganov's assumption was wrong. In [4], it is also proposed another method of solving the problem for the case $n = 2$ which gives some numerical estimates.

In [5] it is proved that in order that collection of mutually non-overlapping closed (open) balls of \mathbb{R}^n , $n \geq 2$ which do not hold the chosen point of \mathbb{R}^n generate the shadow at the point, it is necessary and sufficient to have n balls.

All results mentioned above and some others related to shadow problem can also be found in the survey article [3].

The following question is naturally arising. What can be done in this problem if we will consider other surfaces instead of sphere? In [7]

(cf., e.g. [6]) there were considered prolate spheroids with value of ratio between its major and minor axis more then $2\sqrt{2}$. Authors proved that in order that the union of mutually non-overlapping closed (open) balls which do not hold the ellipsoid center and with centers on the ellipsoid generate the shadow at the ellipsoid center, it is necessary and sufficient to have three balls. But the question still was opened for the other prolate ellipsoids. In the present work the shadow problem for every domain and every its point is solved, which also gives the answer for the ellipsoids.

Statement of Problem

Lemma 1. *Let two open or closed non-overlapping balls of \mathbb{R}^n , with centers on a sphere S^{n-1} and with radiuses less than the sphere radius, are given. Then, every ball with center outside of the sphere, homothetic to the smaller ball (homothetic to one of the balls, if they are equal) relative to the sphere center, would not intersect the other ball.*

Proof. Without loss of generality, let us consider the unit sphere. We denote the balls as $B_1(O_1, R)$, $B_2(O_2, r)$ with radiuses R, r and centers O_1, O_2 respectively and let

$$r \leq R. \quad (1)$$

Let the distance between the balls' centers is equal to d , then, the condition such that the balls are non-overlapping is equivalent to the following inequality

$$r + R \leq d. \quad (2)$$

Further, we will consider the angle α (figure 1) between the segment OO_2 joining the sphere center and the center of the ball B_2 and the ray coming out of the sphere center and which is tangent to the ball B_2 at the point A . And we will also consider the angle φ between the segment OO_2 and the perpendicular to d . We do not consider the case when the triangle O_1OO_2 is degenerated into the segment O_1O_2 , as it coincides

to diametrically opposite situation of the balls B_1 , B_2 which obeys the conditions of lemma 1. It is easy to see that $\angle\alpha \leq \angle\varphi$. Indeed, let us consider right triangles OAO_2 and ODO_2 . We get from them $\sin \alpha = r$, $\sin \varphi = \frac{1}{2}d$. The inequalities (1), (2) give $r \leq \frac{1}{2}d$, that is why

$$\sin \alpha \leq \sin \varphi,$$

which leads to $\angle\alpha \leq \angle\varphi$.

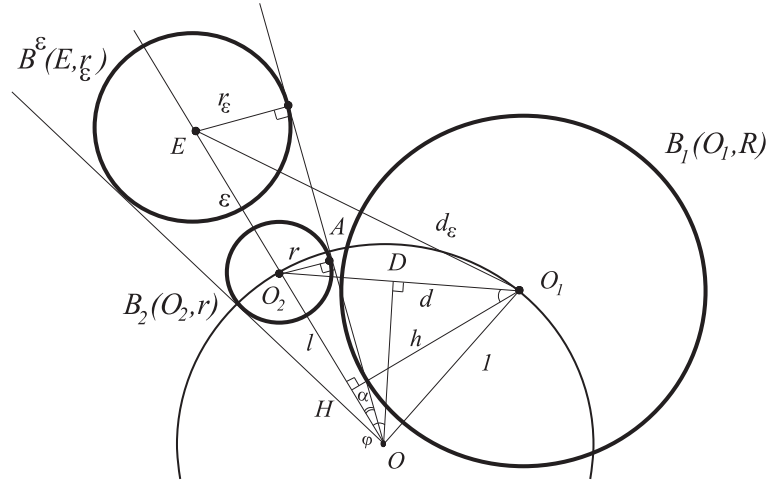


Figure 1

Now let us consider a ball $B^\epsilon(E, r_\epsilon)$ with the center outside of the sphere at the point E , situated at the distance ϵ from the center of the ball B_2 , with radius r_ϵ and which is homothetic to the ball B_2 relative to the sphere center O . Let us show that for this ball it is also valid the inequality

$$r_\epsilon + R \leq d_\epsilon,$$

where d_ϵ be the distance between the centers of the balls B^ϵ , B_1 .

First we notice that the equality $r_\epsilon = r + \epsilon \sin \alpha$ and (2) give

$$r_\epsilon + R \leq d + \epsilon \sin \alpha.$$

Further, we drop the altitude $O_1H = h$ in the triangle $\triangle O_1OO_2$ from the vertex O_1 , then $\angle O_2O_1O = \varphi$ and

$$HO_2 = l = d \sin \varphi.$$

From $\triangle O_1HO_2$ we get

$$d^2 = h^2 + l^2,$$

and from $\triangle O_1HE$ and two privies formulas we have

$$d_\varepsilon^2 = h^2 + (l + \varepsilon)^2 = d^2 + \varepsilon^2 + 2l\varepsilon = d^2 + \varepsilon^2 + 2d\varepsilon \sin \varphi. \quad (3)$$

Let us consider the formula Розглянемо вираз

$$(d + \varepsilon \sin \alpha)^2 = d^2 + \varepsilon^2 \sin^2 \alpha + 2d\varepsilon \sin \alpha. \quad (4)$$

Now from (3), (4) it is easy to see that

$$d + \varepsilon \sin \alpha \leq d_\varepsilon,$$

so, $r_\varepsilon + R \leq d_\varepsilon$, which was to be demonstrated. Lemma 1 is proved.

It is not difficult to give a counter example with balls, homothetic to the smaller ball relative to the sphere center and with centers inside the sphere.

Corollary 1. *Let two open or closed non-overlapping balls of \mathbb{R}^n , with centers on a sphere S^{n-1} and with radiuses less than the sphere radius, are given. Then, every ball with center inside of the sphere, homothetic to the bigger ball (homothetic to one of the balls, if they are equal) relative to the sphere center, would not intersect the other ball.*

Corollary 2. *Let two open or closed non-overlapping balls of \mathbb{R}^n , with centers on a sphere S^{n-1} and with radiuses less than the sphere radius, are given. Then, every ball, homothetic to the smaller ball (homothetic to one of the balls, if they are equal), relative to the sphere center, with coefficient of homothety k_1 would not intersect every ball, homothetic to*

the bigger one, relative to the sphere center, with coefficient of homothety k_2 , if $k_1 \geq k_2$.

It was proved in [4] that it is possible to generate the shadow in the center of sphere S^{n-1} by four given balls of \mathbb{R}^n , proposing one of the placement methods of their centers on the sphere. In the following lemma we give our method, convenient for further statements. We will also consider double-napped cones with common vertex at the center of the sphere with generators tangent to the balls. Here and further such cones will be referred to as *cones under the balls*.

Lemma 2. *There are four closed (opened) non-overlapping balls of \mathbb{R}^n , with centers in the vertexes of regular triangular pyramid and with radii less than the radius of sphere circumscribed about the pyramid such that generate the shadow in the sphere center.*

Proof. Without loss of generality, we will consider the unit sphere S^2 which center for convenience will be placed at the point $O_1 = (0, -1, 0)$ (Figure 2). Let us place the center of the unit open ball $B_1(O, 1)$ at the origin, $O = (0, 0, 0)$. Then the ball B_1 will generate shadow for the sphere center everywhere except sphere equatorial plane $\Sigma = \{(x, y, z) : y = -1\}$ tangent to the ball B_1 at the point O_1 . Let us establish dependence of the angle φ at which the intersection of the plane Σ with every ball $B^X(X, r(X))$ tangent to the ball B_1 with center at point $X \in \mathbb{R}^3$ and radius r can be seen from the point O_1 , on the point X .



Without loss of generality, let us consider opened balls B^X with centers at points $X = (x, y, 0)$ of the half-plane xOy , $x > 0$, and such that have nonempty intersection with plane Σ . Then this intersection will be an opened ball with center at a point $(x, -1, 0)$ and some radius r_0 . We obtain:

$$\begin{aligned}\sin \frac{\varphi}{2} &= \frac{r_0}{x}, \\ r_0 &= \sqrt{r^2 - (-y - 1)^2}, \\ r &= \sqrt{x^2 + y^2} - 1.\end{aligned}$$

Dropping intermediate calculations, we get:

$$\begin{aligned}\sin \frac{\varphi}{2} &= \frac{\sqrt{x^2 - 2y - 2\sqrt{x^2 + y^2}}}{x}, \quad x > 0. \\ \frac{1}{4}x^2 \cos^2 \frac{\varphi}{2} - \frac{1}{\cos^2 \frac{\varphi}{2}} &= y.\end{aligned}\tag{5}$$

Further, we will consider among all balls B^X those, whose centers are on the sphere. The following system is obtained:

$$\begin{cases} \sin \frac{\varphi}{2} = \frac{\sqrt{x^2 - 2y - 2\sqrt{x^2 + y^2}}}{x}, \\ x^2 + (y + 1)^2 = 1, \\ x > 0, \end{cases}$$

that is equivalent to equation

$$\sin^2 \frac{\varphi}{2} = \frac{-y^2 - 4y - 2\sqrt{-2y}}{-y^2 - 2y}, \quad \text{при } -2 < y < 0,\tag{6}$$

or to equation

$$\cos^2 \frac{\varphi}{2} y^2 + 2 \left(\cos^2 \frac{\varphi}{2} + 2 \right) y + \frac{4}{\cos^2 \frac{\varphi}{2}} = 0 \quad \text{при } -2 < y < 0.$$

Let us find the value of the coordinate y of the center of the ball B^X that overlaps $\angle \varphi = \frac{\pi}{3}$. We obtain:

$$y' = \frac{1}{3}(\sqrt{57} - 11) \approx -1,15.$$

At this point

$$x' = \frac{4}{3}\sqrt{\sqrt{57} - 7} \approx 0,99.$$

Thus, every ball, tangent to the ball B_1 with center on the sphere at the level $y = y'$, will overlap a sector with $\angle \varphi = \frac{\pi}{3}$ in the plane Σ . Herewith,

it is easy to see that radius of such ball $r' = \sqrt{(x')^2 + (y')^2} - 1 \approx 0,52$ is less than a half of side $a = \sqrt{3}x' \approx 1,71$ of regular triangle inscribed in circumference on the sphere in the plane $\Pi = \{(x, y, z) : y = y'\}$. Thus, three balls, tangent to the ball B_1 with centers at the vertexes of the triangle, will not overlap.

Now, let us consider plane $\Pi_\varepsilon = \{(x, y, z) : y = y' - \varepsilon\}$. We place the centers of three balls tangent to the ball B_1 at the vertexes of a regular triangle inscribed in circumference that is the intersection of plane Π_ε with sphere. At this point, we choose $\varepsilon > 0$ small enough for these balls to not intersect yet. Since the function $\sin^2 \frac{\varphi}{2}$ in (6) is decreasing, then every such a ball will overlap sector $\angle \varphi > \frac{\pi}{3}$ in the plane Σ . Thus, altogether they will generate the shadow at the sphere center but only in the plane Σ and in the surface of every cone with vertex at the point O_1 and with apex angle close enough to π . Now, we will decrease the radius of the ball B_1 in order that the angle of the cone under it remain close enough to π . And, thus, we obtain set of balls satisfying conditions of the lemma 2.

The lemma 2 is proved.

In [2], there are considered sets similar to the convex ones which are defined as follows.

Definition 1. ([2]) *The set $E \subset \mathbb{R}^n$ is m -convex relative to the point $x \in \mathbb{R}^n \setminus E$, if there exists an m -dimensional plane L such that $x \in L$ and $L \cap E = \emptyset$.*

Definition 2. ([2]) *The set $E \subset \mathbb{R}^n$ is m -convex if it is m -convex relative to every point $x \in \mathbb{R}^n \setminus E$.*

It is easy to verify that both definitions satisfy the axiom of convexity: the intersection of every subfamily of such sets also satisfies the definition. For every set $E \subset \mathbb{R}^n$, it can be considered the minimal m -convex set containing E which is called *m -hull* of the set E .

Now, the shadow problem can be considered as a partial case of the

membership of a point to the 1-hull of a union of some collection of balls.

Further, let domain be an open connected set.

Theorem 1. *In order that every given point x_0 of a domain $D \subset \mathbb{R}^2$ belong to 1-hull of mutually non-overlapping closed (open) balls which do not hold the point x_0 and with centers on the boundary of the domain D , it is sufficient to have two balls.*

Proof. Let us consider the circumference S^1 with center at the point x_0 and with maximal radius such that its interior is contained in domain D . Then, let us construct the collection of two mutually non-overlapping closed (open) circles which do not hold the point x_0 and with centers on the circumference as it was done in [8]. Along with, let us situate the center of the bigger circle at the point $x \in \partial D \cap S^1$. Now, we construct the circle that is homothetic to the smaller one relative to the point x_0 and with center on ∂D . Then, due to lemma 1, this ball do not intersect the bigger one. The theorem is proved.

Theorem 2. *In order that every given point x_0 of a domain $D \subset \mathbb{R}^3$ belong to 1-hull of mutually non-overlapping closed (open) balls which do not hold the point x_0 and with centers on the boundary of the domain D , it is sufficient to have four balls.*

Proof. Let us consider sphere S^2 with center at the point x_0 and with maximal radius such that its interior is contained in the domain D . Then, let us construct the collection of four mutually non-overlapping closed (open) balls which do not hold the point x_0 and with centers on the sphere as it was done in lemma 2. Along with, let us situate the center of the biggest ball at the point $x \in \partial D \cap S^2$. Now, we use the homothety to the rest three balls relative to the point x_0 in such a way as the centers of homothetic balls to be on ∂D . Then, due to lemma 1 each of these balls would not intersect the biggest one and due to corollary 2 these balls would not intersect each other. The theorem is proved.

There exist domain of \mathbb{R}^3 with point that belongs to 1-hull of three

mutually non-overlapping closed (open) balls which do not hold the point and with centers on the boundary of the domain. A prolate ellipsoid with value of ratio between its major and minor axis more than $2\sqrt{2}$ is the example of such domain and the ellipsoid center is the point ([6], [7]).

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